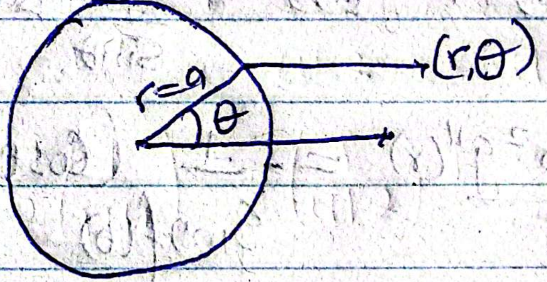
MATH 414:

Motion of Sphere in an infinite mass of fluid

Case I:

Consider the origin o at the centre of the sphere with radius $r=a$ which is moving along its axis. Determine velocity potential, stream-function, equation of streamline and complex potential.

soln



Laplace eqn for velocity potential in spherical is given by:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right)+\frac{\sin\theta}{r^2}\frac{\partial^2\phi}{\partial z}=0$$

By symmetry $\frac{\sin\theta}{r^2}\frac{\partial^2\phi}{\partial\theta}=0$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right)=0$$

let $\phi(r,\theta) = g(r) f(\theta)$

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}(g(r) f(\theta))\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial(g(r) f(\theta))}{\partial\theta}\right)$$

$$f(\theta)\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} g(r)\right] + \frac{g(r)}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial f(\theta)}{\partial\theta}\right]$$

$$f(\theta)\left[2r g'(r) + r^2 g''(r)\right] + \frac{g(r)}{\sin\theta}\left[\cos\theta f'(\theta) + \sin\theta f''(\theta)\right]$$

$$f(\theta)\left[2r g'(r) + r^2 g''(r)\right] = -\frac{g(r)}{\sin\theta}\left[\cos\theta f'(\theta) + \sin\theta f''(\theta)\right]$$

$$\frac{2r g'(r) + r^2 g''(r)}{g(r)} = -\frac{1}{\sin\theta f(\theta)}\left(\cos\theta f'(\theta) + \sin\theta f''(\theta)\right) = \alpha$$

$\therefore$

$$\frac{2r g'(r) + r^2 g''(r)}{g(r)} = \alpha.$$

$$2r g'(r) + r^2 g''(r) - \alpha g(r) = 0 \quad \text{— (1)}$$

$$r^2 g''(r) + 2r g'(r) - \alpha g(r) = 0 \quad \text{——— (i)}$$

Also,

$$-\left[\cos\theta f'(\theta) + \sin\theta f''(\theta)\right] = \alpha \sin\theta f(\theta)$$

$$\sin\theta f''(\theta) + \cos\theta f'(\theta) + \alpha \sin\theta f(\theta) = 0 \quad \text{——— (ii)}$$

from (1) Let: $g(r) = r^m$ $g'(r) = m r^{m-1}$ $g''(r) = m(m-1)r^{m-2}$

$\therefore$

$r^2 (m)(m-1) r^{m-2} + 2r\, m r^{m-1} - \alpha r^m = 0$

$m(m-1) r^m + 2m r^m - \alpha r^m = 0$

$m(m-1) + 2m - \alpha = 0$

$m^2 - m + 2m - \alpha = 0$

$m^2 + m - \alpha = 0$ Let $\alpha = n(n+1) = n^2 + n$

$m^2 + m - n^2 - n = 0$

$m^2 - n^2 + m - n = 0$

$(m+n)(m-n) + 1(m-n) = 0$

$m - n = 0$ $m + n + 1 = 0$

$m = n$ $m = -(n+1)$

$\therefore$

$g(r) = Ar^n + Br^{-(n+1)}$

$g(r) = Ar^n + \dfrac{B}{r^{n+1}}$

Let $n = 1$

$\therefore$

$g(r) = Ar + \dfrac{B}{r^2}$

$\therefore$

Similarly:

$\sin\theta f''(\theta) + \cos\theta f'(\theta) + \alpha \sin\theta f(\theta) = 0$

Let $x = \cos\theta$; $\frac{dx}{d\theta} = -\sin\theta$

$$\frac{df(\theta)}{d\theta} = \frac{d\theta}{dx}\left(\frac{dx}{d\theta}\right) = \frac{d\theta}{dx}(-\sin\theta) = -\sin\theta\frac{d\theta}{dx}$$

$$\frac{d^2 f(\theta)}{d^2\theta} = \frac{d}{d\theta}\left(-\sin\theta\frac{d\theta}{dx}\right)$$

$$= -\left[\sin\theta\frac{d^2 f(\theta)}{dx^2}\frac{dx}{d\theta} + \cos\theta\frac{d\theta}{dx}\right]$$

$$= -\left[\sin\theta\frac{dx}{d\theta}\left(\frac{d^2 f(\theta)}{dx}\right) + \cos\theta\frac{df(\theta)}{dx}\right]$$

$$= -\left[\sin\theta(-\sin\theta)\frac{d^2 f(\theta)}{dx} + \cos\theta\frac{df(\theta)}{dx}\right]$$

$$= \sin^2\theta\frac{d^2 f(\theta)}{dx^2} - \cos\theta\frac{df(\theta)}{dx}.$$

$$\frac{d^2 f(\theta)}{d^2\theta} = (1-\cos^2\theta)\frac{d^2 f(\theta)}{dx^2} - \cos\theta\frac{df(\theta)}{dx}.$$

put all in ②

$$\sin\theta\left[(1-\cos^2\theta)\frac{d^2 f(\theta)}{dx^2} - \cos\frac{df(\theta)}{dx}\right] + \cos\theta\left(-\sin\theta\frac{d\theta}{dx}\right)$$

$$+ \alpha\sin\theta f(\theta) = 0$$

$\therefore$

$$[(1-\cos^2\theta)f''(\theta) - \cos f'(\theta) - \cos\theta f'(\theta) + \alpha f(\theta) =$$

$(1-\cos^2\theta)f''(\theta) - 2\cos\theta f'(\theta) + \alpha f(\theta) = 0$

since $x = \cos\theta$

$(1-x^2)f''(\theta) - 2x f'(\theta) + \alpha f(\theta) = 0$

let $\alpha = n(n+1)$

$(1-x^2)f''(\theta) - 2x f'(\theta) + n(n+1) f(\theta) = 0$

Using Rodrigue's formula:

$P_n(x) = \cancel{\text{Rodrigue}}\ \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

$P_0(x) = 1, \quad P_1(x) = x,$

if $n=1$, $P_1(\cancel{\cos\theta})\ \cos\theta = f(\theta)$.

$\therefore$

$\phi(r,\theta) = g(r) f(\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta.$

$\therefore$

Case i:

$\frac{\partial \phi}{\partial r} = -V\cos\theta \qquad \frac{\partial \phi}{\partial r} = 0$

at $r = a \qquad\qquad r \to \infty.$

$$\frac{\partial}{\partial r}\left[Ar + \frac{B}{r^2}\right]\cos\theta = -V\cos\theta \qquad r \to a$$

$$\left(A - \frac{2B}{a^3}\right)\cancel{\cos\theta} = -V\cancel{\cos\theta}$$

$$\therefore \quad -V = A - \frac{2B}{a^3} \longrightarrow (1)$$

$\therefore$

$$\frac{\partial}{\partial r}\left[Ar + \frac{B}{r^2}\right]\cos\theta = 0 \qquad r \to \infty.$$

$$\left(A - \frac{2B}{\infty^3}\right)\cos\theta = 0$$

$$A\cos\theta = 0 \Rightarrow A = 0.$$

from (1)

$$-V = 0 - \frac{2B}{a^3} \Rightarrow V = \frac{2B}{a^3}.$$

$$B = \frac{a^3 V}{2}.$$

$$\phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta$$

$$\phi(r,\theta) = \frac{a^3 V\cos\theta}{2r^2}$$ — Velocity potential

$\frac{1}{r^2} = r^{-2}$

$-2r^{-3}$

ii Stream function: $\frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$

$\therefore \frac{\partial \psi}{\partial \theta} = r \frac{\partial \phi}{\partial r} = \cancel{r \frac{(A-2B)}{r^3}\cos\theta}.$

$= r \left(\frac{-2a^3 V \cos\theta}{2r^3} \right)$

$\frac{\partial \psi}{\partial \theta} = -\frac{a^3 V \cos\theta}{r^2}$

$\therefore$

$\partial \psi = -\frac{a^3 V \cos\theta}{r^2} \partial \theta.$

integrate both side w.r.t θ.

$\psi(r, \theta) = -\frac{a^3 V \sin\theta}{r^2} + f(r)$ — stream fn

iii The complex potential:

$Z = \phi + i\psi$

$Z = \frac{a^3 V}{2r^2} \cos\theta - \left(\frac{a^3 V \sin\theta}{r^2} + h(r) \right) i$

iv Equation of Streamlines:

$$\frac{dr}{\frac{\partial \phi}{\partial r}} = \frac{r\,d\theta}{\frac{1}{r}\frac{\partial \psi}{\partial \theta}}$$

$\underline{d\phi}$ $\phi(r,\theta) = \frac{a^3 V}{\partial r^2} \cos\theta$

$\psi(r,\theta) = -\frac{a^3 \sin\theta}{r^2} + h(r)$

$\frac{\partial \psi}{\partial r} = -\frac{a^3 V \cos\theta}{r^3}$

$\frac{\partial \psi}{\partial \theta} = \frac{-a^3 V \cos\theta}{r^2}$

$\therefore$ Eqn of S.L

$\frac{dr}{\frac{\partial \phi}{\partial r}} = \frac{r\,d\theta}{\frac{1}{r}\frac{\partial \psi}{\partial \theta}} \Rightarrow \frac{dr}{-\frac{a^3 V \cos\theta}{r^3}} = \frac{r\,d\theta}{\frac{1}{r}\left(\frac{-a^3 V \cos\theta}{r^2}\right)}$

$\Rightarrow \frac{dr}{\cancel{\frac{-a^3 V \cos\theta}{r^3}}} = \frac{r\,d\theta}{\cancel{\frac{-a^3 V \cos\theta}{r^3}}}$

$\Rightarrow dr = r\,d\theta$

$\frac{dr}{r} = d\theta$

integrating both side

$\ln r = \theta + c$

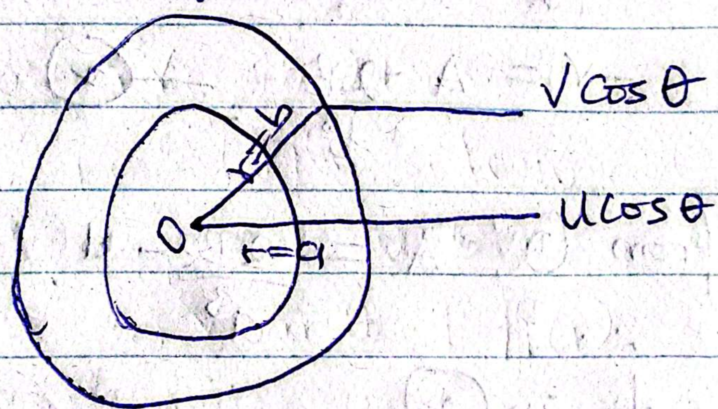
$r = e^{\theta + c}$

$r = e^c \cdot e^\theta = Ae^\theta$

$\therefore$

$r = Ae^\theta$, where $A = e^c$.

Case II: Motion of a Concentric sphere in a liquid
Consider the motion Concentric sphere of radius a and b such $a<b$. Moving in an infinite mass of sphere where the inner and outer move along the x-axis with uniform velocity of u and v respectively. Find the velocity potential, stream function, complex potential and ~~eq~~ equation of stream lines:



from the result in Case I:

$$\phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta .$$

$$\frac{\partial \phi}{\partial r} = -u\cos\theta \quad r=a$$

$$\frac{\partial \phi}{\partial r} = -v\cos\theta \quad r=b .$$

$$\frac{\partial}{\partial r}\left[\left(Ar+\frac{B}{r^2}\right)\cos\theta\right] = -u\cos\theta \quad r=a$$

$$\left(A-\frac{2B}{a^3}\right)\cancel{\cos\theta} = -u\cancel{\cos\theta}$$

$$-u = A-\frac{2B}{a^3} \Rightarrow \cancel{-u = \frac{a^3A - 2B}{a^3}} \quad \text{—— (1)}$$

Similarly:

$$A-\frac{2B}{b^3} = -v$$

$$-v = A-\frac{2B}{b^3} \quad \text{—— (2)}$$

from (1), $A = \frac{2B}{a^3} - u$

put in (2)

$$-v = \frac{2B}{a^3} - u - \frac{2B}{b^3}$$

$$u-v = \frac{2B}{a^3} - \frac{2B}{b^3}$$

$$u-v = \frac{2B(b^3-a^3)}{a^3b^3}$$

$$B = \frac{(u-v)a^3b^3}{2(b^3-a^3)}$$

$$B = \frac{(v-u)a^3b^3}{2(a^3-b^3)}$$

$\therefore$ ~~$\phi(r,\theta) =$~~

$$A = \frac{2}{\cancel{a^3}}\left[\frac{(V-U)\cancel{a^3}b^3}{2(a^3-b^3)}\right] - u$$

$$A = \frac{\cancel{2}(V-u)b^3}{\cancel{2}(a^3-b^3)} - u = \frac{(V-u)b^3 - u(a^3-b^3)}{a^3-b^3} \quad : a<b.$$

$\therefore$

$$\phi(r,\theta) = \left[\frac{(V-u)b^3 - u(a^3-b^3)}{a^3-b^3} r + \frac{(V-u)a^3b^3}{2r^2(a^3-b^3)}\right]\cos\theta$$

$\therefore$ ~~Stream~~ function: $\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$.

$$\frac{\partial\psi}{\partial\theta} = \frac{r\,\partial\phi}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = r\left[\frac{(V-u)b^3 - u(a^3-b^3)}{a^3-b^3} - \frac{(V-u)a^3b^3}{r^3(a^3-b^3)}\right]\cos\theta.$$

$$\partial\psi = r\left[\frac{(V-u)b^3 - u(a^3-b^3)}{a^3-b^3} - \frac{(V-u)a^3b^3}{r^3(a^3-b^3)}\right]\cos\theta\,\partial\theta$$

Integrating both side:

$$\psi(r,\theta) = \left[\frac{(V-u)b^3 - u(a^3-b^3)}{a^3-b^3} r - \frac{(V-u)a^3b^3}{r^2(a^3-b^3)}\right]\sin\theta + h(r).$$

Complex potential:

$\chi = \phi + i\psi$

$$\chi = \left[\frac{(V-U)b^3 - U(a^3-b^3)r}{(a^3-b^3)} + \frac{(V-U)a^3b^3}{2r^2(a^3-b^3)}\right]\cos\theta + \cancel{i\sin\theta}$$

$$i\left[\frac{(V-U)b^3 + U(a^2-b^3)r}{a^3-b^3} \cancel{+} \frac{(V-U)a^3b^3}{r^2(a^3-b^3)}\right]\cancel{\sin\theta} + i\sin\theta$$

Equation of streamlines

$$\frac{dr}{\frac{\partial\phi}{\partial r}} = \frac{r\,d\theta}{\frac{1}{r}\frac{\partial\psi}{\partial\theta}}$$

$$\frac{\partial\phi}{\partial r} = \left[\frac{(V-U)b^3 - U(a^3-b^3)}{a^3-b^3} - \frac{(V-U)a^3b^3}{r^3(a^3-b^3)}\right]\cos\theta = A$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\left[\frac{(V-U)b^3 - U(a^3-b^3)r}{a^3-b^3} \cancel{+} \frac{(V-U)a^3b^3}{r^3(a^2-b^3)}\right]\cos\theta + 0$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \left[\frac{(V-U)b^3 - U(a^3-b^3)}{a^3-b^3} + \frac{(V-U)a^3b^3}{r^3(a^3-b^3)}\right]\cos\theta = A$$

$\therefore$

$$\frac{dr}{\cancel{A}} = \frac{r\,d\theta}{\cancel{A}} \Rightarrow dr = r\,d\theta$$

$$\frac{dr}{r} = d\theta$$

Integrate both side.

$\ln r = \theta + c$

$r = e^{\theta + c} \Rightarrow r = e^{\theta} \cdot e^{c}$

CamScanner

$r = \cancel{B}e^{\theta}$ $\quad B = e^{c}$.

$\therefore$

Eqn of S.L $\Rightarrow r = Be^{\theta}$.

Flow

In fluid Dynamics, flow is the movement of a fluid (liquid or gas) from one place to another, driven by forces such as pressure differences, gravity, or moving boundaries. It is essentially how and why a fluid moves.

When we talk about flow, we describe:

- Direction (where the fluid is going)
- speed/velocity (how fast it moves)
- pattern (smooth, chaotic, Layered etc)
- causes (pressure, temp, mechanical motion).

Kinds of flow

1. Steady flow: A flow is said to be steady if the flow parameters such as velocity, temperature acceleration and pressure are time independent, otherwise, if they are time dependent is then called unsteady flow (transient). eg water flowing at a constant rate through a straight garden hose; filling an emptying water tank

2. Fully developed: A flow is said to fully developed if the velocity and temperature are unidirectional while the entry region is called developed flow.
eg water far downstream in along straight pipe; water just entering a pipe from a reservoir

3. Laminar flow: It is a gentle and smooth flow with low velocity. eg honey flowing slowly through bottle.

4. Turbulent flow, is chaotic flow with high velocity and high stability of each velocity.
eg smoke rising from a candle after it becomes unstable; water from a fast-flowing tap.

5. Couette: It is a flow driven by the movement of a boundary surface, not by a pressure difference. It is a flow induced by the motion of at least one of the plate.
eg fluid between two parallel plates where one plate moves & drags the fluid with it (like in lubrication)

6. Pressure-driven flow: ~~it is~~ a flow induced by the result or by application of constant pressure

7. Natural Convection flow: It is a flow induced by buoyancy forces caused by temperature difference within the fluid. e.g warm air rising near a heater.

8. Irrotational flow: ~~A~~ flow in which fluid particles do not rotate about their own axes as they move. it is also ~~say~~ said to be an irrotational flow If the curl of a vector $\vec{u} = 0$. (vorticity $(w) = 0$). e.g flow ~~of~~ air over an airplane wing. or flow toward the end of straw.

9. Rotational flow: A flow in which fluid particles rotate about their own axes as they move. When a flow is said to be rotational it means that the curl $\vec{u} \neq 0$ or vorticity is not zero every where in the flow.

1. Laplacian in Cartessian Coordinates

For a scalar field $\phi(x, y, z)$:

$$\nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

if $\nabla^2\phi = 0$;

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

2. Laplacian in Cylindrical Coordinates (r, θ, z)

$r \Rightarrow$ radial distance from z-axis

$\theta \rightarrow$ azimuthal angle

$z \Rightarrow$ same as cartessian z

Laplacian eqn:

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \phi}{\partial \phi^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Laplace:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

3. Spherical Coordinates (r, θ, φ)

r = distance from origin

θ = polar angle

· φ - azimuthal angle: (in xy-plane from x-axis)

Laplacian:

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) +$$

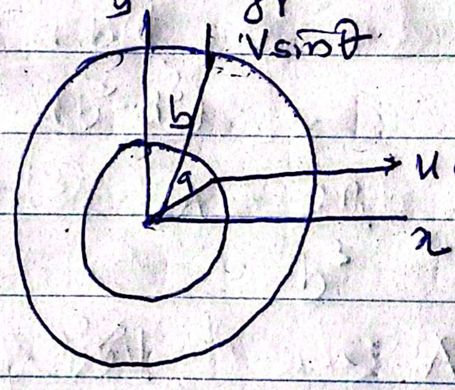
$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

Laplace Eqn:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

3- Case iv

Consider motion of two concentric spheres with a and b where $a<b$ moving in an infinite mass of fluid where the inner sphere moves along x-axis with uniform velocity u and outter the sphere moves perpendicular to the x-axis uniform velocity V. Find the velocity potential, the streaming function, the eqn of streamlines and the magnitude of velocity and complex velocity $\left(Z = \frac{\partial \phi}{\partial r} + i \frac{\partial \psi}{\partial \theta}\right)$.



soln

Since the flow is moving along x and y axis implies that the flow is possible.

$\nabla \cdot u = 0$.

$\nabla \times u = 0 \Rightarrow \exists\, \phi \ni u = \nabla\phi$. This implies that the Laplacian eqn is satisfied:

$\nabla^2 \phi = 0$.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0.$$

So, the functn

$$\phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta + \left(Cr + \frac{D}{r^2}\right)\sin\theta$$

Subject to:

$$\frac{\partial \phi}{\partial r} = -U\cos\theta \quad \text{at } r = a$$

$$\frac{\partial \phi}{\partial r} = -V\sin\theta \quad \text{at } r = b$$

1st condn:

$$\frac{\partial}{\partial r}\left[\left(Ar + \frac{B}{r^2}\right)\cos\theta + \left(Cr + \frac{D}{r^2}\right)\sin\theta\right] = -U\cos\theta$$

$$\left(A - \frac{2B}{a^3}\right)\cos\theta + \left(C - \frac{2D}{a^3}\right)\sin\theta = -U\cos\theta \quad \text{--- } (*)$$

2nd condn:

$$\frac{\partial}{\partial r}\left[\left(Ar + \frac{B}{r^2}\right)\cos\theta + \left(Cr + \frac{D}{r^2}\right)\sin\theta\right] = -V\sin\theta \quad r = b.$$

$$\left(A - \frac{2B}{b^3}\right)\cos\theta + \left(C - \frac{2D}{b^3}\right)\sin\theta = -V\sin\theta \quad \text{--- } (**)$$

from $(*)$ Equate the coefficient of $\cos\theta$; \$ $\sin\theta$

$$A - \frac{2B}{a^3} = -U \quad \text{--- } (3) \quad \Rightarrow \quad A = \frac{2B}{a^3} - U \quad \text{--- } (B)$$

$$C - \frac{2D}{a} = 0 \quad \Rightarrow \quad C = \frac{2D}{a} \quad \text{--- } (g)$$

from (**), Equating coefficient of $\sin\theta$ & $\cos\theta$

$$A - \frac{2B}{b^3} = 0 \Rightarrow (3)$$

$$C - \frac{2D}{b^3} = -V \quad \text{---} (4)$$

from (1) & (3)

$$\frac{2B}{a^3} - u - \frac{2B}{b^3} = 0$$

$$2B\left(\frac{b^3 - a^3}{a^3 b^3}\right) = u$$

$$\boxed{B = \frac{a^3 b^3 u}{2(b^3 - a^3)}}$$

from (3): $A = \frac{2B}{b^3}$.

$$A = \frac{\frac{\cancel{2}\, a^3 \cancel{b^3} u}{\cancel{2}(b^3 - a^3)}}{\cancel{b^3}} \Longrightarrow \boxed{A = \frac{a^3 u}{b^3 - a^3}}$$

Also; from (2) & (4)

$$C = \frac{2D}{a^3} \Longrightarrow \frac{2D}{a^3} - \frac{2D}{b^3} = -V$$

$$2D\frac{(b^3 - a^3)}{a^3 b^3} = -V$$

$$\boxed{D = \frac{-a^3 b^3 V}{2(b^3 - a^3)}}$$

$$C = \frac{2\Delta}{a^3} = \frac{\cancel{2}\left(\frac{-a^3b^3V}{\cancel{2}(b^3-a^3)}\right)}{\cancel{a^3}}$$

$$\boxed{C = \frac{-b^3V}{(b^3-a^3)}}$$

(i) The velocity potential becomes:

$$\phi(x,y) = \left[\frac{a^3ur}{(b^3-a^3)} + \frac{a^3b^3u}{2r^2(b^3-a^3)}\right]\cos\theta + \left[\frac{b^3Vr}{b^3-a^3} + \frac{a^3b^3V}{2r^2(b^3-a^3)}\right]\sin\theta$$

(ii) Stream function: $\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$

$$\frac{\partial\psi}{\partial\theta} = r\frac{\partial\phi}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = r\left[\frac{a^3u}{(b^3-a^3)} - \frac{a^3b^3u}{r^3(b^3-a^3)}\right]\cos\theta - \left[\frac{b^3Vr}{(b^3-a^3)} - \frac{a^3b^3V}{r^3(b^3-a^3)}\right]\sin\theta$$

$$\partial\psi = \left[\frac{a^3ur}{(b^3-a^3)} - \frac{a^3b^3u}{r^2(b^3-a^3)}\right]\cos\theta - \left[\frac{b^3Vr}{(b^3-a^3)} - \frac{a^3b^3V}{r^2(b^3-a^3)}\right]\sin\theta \, d\theta$$

Integrate both side:

$$\psi(r,\theta) = \left[\frac{a^3ur}{b^3-a^3} - \frac{a^3b^3u}{r^2(b^3-a^3)}\right]\sin\theta + \left[\frac{b^3Vr}{(b^3-a^3)} - \frac{a^3b^3V}{r^2(b^3-a^3)}\right]\cos\theta$$

$$+ h(r).$$

~~$\phi(r,\theta)$~~.

Equation of streamlines:

$$\frac{dr}{\frac{\partial\phi}{\partial r}} = \frac{r\,d\theta}{\frac{1}{r}\frac{\partial\psi}{\partial\theta}}$$

$$\frac{\partial\phi}{\partial r} = \left[\frac{a^3u}{b^3-a^3} - \frac{a^3b^3u}{r^3(b^3-a^3)}\right]\cos\theta - \left[\frac{b^3V}{(b^3-a^3)} + \frac{a^3b^3V}{r^3(b^3-a^3)}\right]\sin\theta$$
$= A.$

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\left[\frac{a^3ur}{b^3-a^3} - \frac{a^3b^3u}{r^2(b^3-a^3)}\right]\cos\theta - \frac{1}{r}\left[\frac{a^3vr}{b^3-a^3} + \frac{a^3b^3V}{r^2(b^3-a^3)}\right]\sin\theta$$

$$= \left[\frac{a^3u}{b^3-a^3} - \frac{a^3b^3u}{r^3(b^3-a^3)}\right]\cos\theta - \left[\frac{a^3V}{b^3-a^3} + \frac{a^3b^3V}{r^3(b^3-a^3)}\right]\sin\theta$$
$= A$

$\therefore$

$$\frac{dr}{A} = \frac{r\,d\theta}{A} \Rightarrow dr = r\,d\theta \Rightarrow \int\frac{dr}{r} = \int d\theta$$

Integrate both sides:

$\ln r = \theta + c$

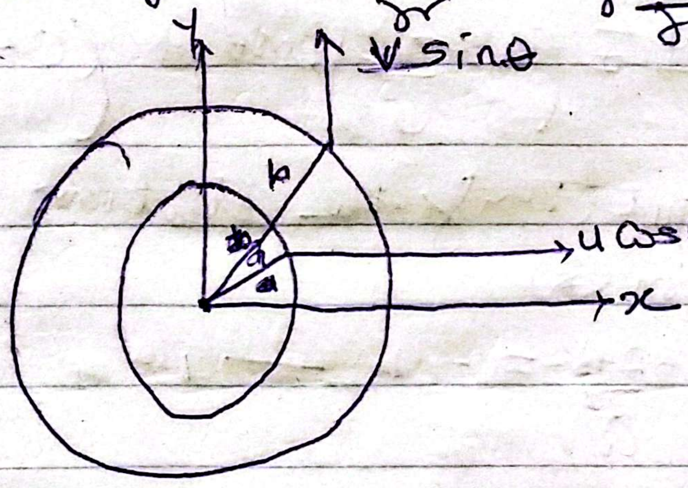
$r = e^{\theta} + e^{c} = e^{\theta}\cdot e^{c}$

$r = Ae^{\theta}$ where $A = e^{c}$

$=$

Case III:

Consider motion of two concentric sphere with a and b where a<b moving in an infinite mass of fluid where the inner sphere moves along x-axis with uniform velocity u and outlet the sphere moves perpendicular to the x-axis uniform velocity v. Find the velocity potential, the stream function, the eqn of streamlines and the magnitudes of velocity and complex velocity $\left(z = \frac{\partial \phi}{\partial r} + i \frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$



Soln

Since the flow is moving along the x & y axis implies that the flow is possible:

$\nabla \cdot u = 0$ & $\nabla \times u = 0$ then there exis ϕ such that $u = \nabla \phi$. This implies that Laplacian equation satisfied: $\nabla^2 \phi = 0$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial \phi}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$

$\therefore$ The soln:

$$\phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta + \left(Cr + \frac{D}{r^2}\right)\sin\theta$$

Subject to:

$$-\frac{\partial\phi}{\partial r} = u\cos\theta \quad \text{at } r = a$$

$$-\frac{\partial\phi}{\partial r} = V\sin\theta \quad \text{at } r = b$$

thus,

$$\frac{\partial\phi}{\partial r} = \left(A - \frac{2B}{a^3}\right)\cos\theta + \left(C - \frac{2D}{a^3}\right)\sin\theta = -u\cos\theta \quad (*)$$

$$\frac{\partial\phi}{\partial r} = \left(A - \frac{2B}{b^3}\right)\cos\theta + \left(C - \frac{2D}{b^3}\right)\sin\theta = -V\sin\theta \quad (**)$$

Equating the Equations:

from $(*)$

$$A - \frac{2B}{a^3} = -u \quad \text{--- i}$$

$$C - \frac{2D}{a^3} = 0 \quad \text{--- ii}$$

$\therefore$ from $**$

$$A - \frac{2B}{b^3} = 0 \quad \text{--- iii}$$

$$C - \frac{2D}{b^3} = -V \quad \text{--- iv}$$

$$\therefore A = \frac{2B}{b^3}$$

$$\therefore \frac{2B}{b^3} - \frac{2B}{a^3} = -u$$

$$\frac{2B(a^3-b^3)}{a^3b^3} = -u \implies B = \frac{-u\,a^3b^3}{2(a^3-b^3)}$$

$$\therefore A = \frac{2}{b^3}\left(\frac{-u\,a^3b^3}{2(a^3-b^3)}\right) = \frac{-u\,a^3}{a^3-b^3}.$$

$\therefore$ from (2).

$$C = \frac{2D}{a^3} \implies \frac{2D}{a^3} - \frac{2D}{b^3} = -V$$

$$\frac{2D(b^3-a^3)}{a^3b^3} = -V \implies D = \frac{Va^3b^3}{2(a^3-b^3)}$$

$\therefore$

$$C = \frac{2}{a^3}\left(\frac{Va^3b^3}{2(a^3-b^3)}\right) = \frac{Vb^3}{a^3-b^3}.$$

$\therefore$

$$A = \frac{-u\,a^3}{a^3-b^3},\quad B = \frac{-u\,a^3b^3}{2(a^3-b^3)},\quad C = \frac{Vb^3}{a^3-b^3},\quad D = \frac{Va^3b^3}{2(a^3-b^3)}$$

(1) $\therefore$ The velocity potential becomes:

$$\phi(r,\theta) = -\left(\frac{ua^3r}{a^3-b^3} + \frac{ua^3b^3}{2r^2(a^3-b^3)}\right)\cos\theta + \left(\frac{Vb^3r}{(a^3-b^3)} + \frac{Va^3b^3}{2r^2(a^3-b^3)}\right)\sin\theta$$

(2) The stream function:

$$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\therefore \frac{\partial \psi}{\partial \theta} = r\frac{\partial \phi}{\partial r}$$

$$\cancel{\frac{\partial \psi}{\partial \theta} = -\left(\frac{a^3 u r^2}{a^3 - b^3} + \frac{u a^3 b^3}{2r^2(a^3-b^3)}\right)\cos\theta + \left(\frac{v b^3 r^2}{a^3-b^3} + \frac{v a^3 b^3}{2r(a^3-b^3)}\right)\sin\theta}$$

$$\frac{\partial \psi}{\partial \theta} = r\left[-\left(\frac{u a^3}{a^3-b^3} + \frac{u a^3 b^3}{r^3(a^3-b^3)}\right)\cos\theta + \left(\frac{v b^3}{a^3-b^3} - \frac{v a^3 b^3}{r^3(a^3-b^3)}\right)\sin\theta\right]$$

$$\frac{\partial \psi}{\partial \theta} = -\left[\frac{u a^3 r}{a^3-b^3} + \frac{u a^3 b^3}{r^2(a^3-b^3)}\right]\cos\theta + \left[\frac{v b^3 r}{a^3-b^3} - \frac{v a^3 b^3}{r^2(a^3-b^3)}\right]\sin\theta$$

$$\partial \psi = \left[-\left(\frac{u a^3 r}{a^3-b^3} + \frac{u a^3 b^3}{r^2(a^3-b^3)}\right)\cos\theta + \left(\frac{v b^3 r}{a^3-b^3} - \frac{v a^3 b^3}{r^2(a^3-b^3)}\right)\sin\theta\right]\partial\phi$$

integrating w.r.t θ gives:

$$\psi(r,\theta) = -\left(\frac{u a^3 r}{a^3-b^3} + \frac{u a^3 b^3}{r^2(a^3-b^3)}\right)\sin\theta + \left(\frac{v b^3 r}{a^3-b^3} - \frac{v a^3 b^3}{r^2(a^3-b^3)}\right)\cos\theta$$

$\therefore$

The streamlines: $\chi = \frac{\partial \phi}{\partial r} + i\frac{1}{r}\frac{\partial \psi}{\partial \theta}$

$\therefore$

$$\chi = -\left(\frac{u a^3}{a^3-b^3} - \frac{u a^3 b^3}{r^3(a^3-b^3)}\right)\cos\theta + \left(\frac{v b^3}{a^3-b^3} - \frac{v a^3 b^3}{r^3(a^3-b^3)}\right)\sin\theta$$

$$+ i\frac{1}{r}\left[-\left(\frac{u a^3 r}{a^3-b^3} - \frac{u a^3 b^3}{r^2(a^3-b^3)}\right)\cos\theta + \left(\frac{v b^3 r}{a^3-b^3} - \frac{v a^3 b^3}{r^2(a^3-b^3)}\right)\sin\theta\right]$$

$$z = -\left(\frac{ua^3}{a^3-b^3} - \frac{ua^3b^3}{r^3(a^3-b^3)}\right)\cos\theta + \left(\frac{vb^3}{a^3-b^3} - \frac{va^3b^3}{r^3(a^3-b^3)}\right)\sin\theta +$$

$$i\left[\left(\frac{-ua^3}{a^3-b^3} + \frac{ua^3b^3}{r(a^3-b^3)}\right)\cos\theta + \left(\frac{vb^3}{a^3-b^3} - \frac{va^3b^3}{r(a^3-b^3)}\right)\sin\theta\right].$$

Equation of streamlines for spherical:

$$\frac{dr}{\frac{\partial\psi}{\partial r}} = \frac{r\,d\theta}{\frac{1}{r}\frac{\partial\psi}{\partial\theta}}$$

$$\therefore \frac{1}{r}\frac{\partial\psi}{\partial\theta}.dr = r\frac{\partial\psi}{\partial r}\,d\theta.$$

$$\int\left[-\left(\frac{ua^3}{a^3-b^3} - \frac{ua^3b^3}{r(a^3-b^3)}\right)\cos\theta + \left(\frac{vb^3}{a^3b^3} - \frac{va^3b^3}{r(a^3-b^3)}\right)\sin\theta\right]dr = \int\left[-\left(\frac{ua^3r}{a^3-b^3} - \frac{ua^3b^3}{r^2(a^3-b^3)}\right)\right.$$

$$\left.\cos\theta + \left(\frac{vb^3}{a^3-b^3} - \frac{va^3b^3}{r^2(a^3-b^3)}\right)\sin\theta\right]d\theta$$

Integrating both side:

The Navier-Stokes Equations:

i. Cartesian Coordinates:

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right] = \vec{F} - \nabla p + \mu\nabla^2\vec{q}$$

$$\rho\left[\frac{\partial (u\hat{i} + v\hat{j} + w\hat{k})}{\partial t} + (u\hat{i} + v\hat{j} + w\hat{k})\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(u\hat{i} + v\hat{j} + w\hat{k})\right.$$

$$= f_x\hat{i} + f_y\hat{j} + f_z\hat{k} - \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)p + \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(u\hat{i} + v\hat{j} + w\hat{k})$$

Along x-axis:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\right].$$

Along y-axis:

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = f_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

Along z-axis:

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = f_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Cylinderical Coordinates :

$$\nabla \cdot u = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$\therefore u(r,\theta,z)$:

Along radal r :

$$\rho\left(\frac{\Delta u_r}{\Delta t} - \frac{u^2 \theta}{r}\right) = \frac{-\partial p}{\partial r} + u\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$+ f_x$$

~~Along~~

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} +$$

$$u\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + \rho f_x .$$

Along θ :

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) =$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) + \rho f_\theta$$

Along z :

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = \frac{-\partial p}{\partial z} + \mu\nabla^2 u_z + \rho f_z$$

Spherical Coordinates: (r, θ, ϕ)

$\therefore U = (U_r, U_\theta, U_\phi)$

Continuity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial u_\phi}{\partial \phi} = 0$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(U_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi} = 0$$

Radial r:

$$\rho\left[\frac{\partial U_r}{\partial t} + U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} + \frac{U_\phi}{r\sin\theta}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi^2 + U_\theta^2}{r}\right]$$

$$= -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 U_r - \frac{2U_r}{r^2} - \frac{2}{r^2}\left(\frac{\partial U_\theta}{\partial \theta} + U_\theta\cot\theta + \frac{1}{\sin\theta}\frac{\partial U_\phi}{\partial \phi}\right)\right]$$

$$+ \rho f_r$$

Polar θ:

$$\rho\left[\frac{\partial U_\theta}{\partial t} + U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_\phi}{r\sin\theta}\frac{\partial U_\theta}{\partial \phi} + \frac{U_r U_\theta}{r}\right.$$

$$\left. - \frac{U_\phi^2\cot\theta}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\nabla^2 U_\theta - \frac{U_\theta}{r^2\sin^2\theta} + \right.$$

$$\left.\frac{2}{r^2}\frac{\partial U_r}{\partial \theta} - \frac{2\cos\theta}{r^2\sin\theta}\frac{\partial U_\phi}{\partial \phi}\right) + \rho f_\theta .$$

Azimuthal ϕ:

$$\rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r}\right.$$

$$\left. + \frac{u_\theta u_\phi}{r}\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left[\nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \right.$$

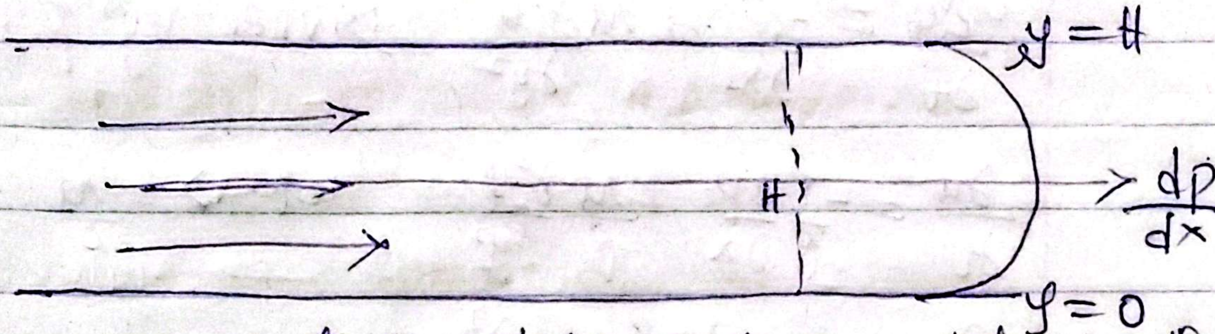
$$\left.\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \phi}\right) + \rho f_\phi.$$

where the scalar Laplacian is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

Exact Solution to the Navier-Stokes Equation

Consider the two dimensional laminar flow of an incompressible fluids of constant viscosity between two parallel at a distance H apart



Assuming that the plate is horizontal, so that H is measured in the vertical direction. Since the flow is two-dimensional it does not vary with z. $\vec{u} = (u, v, 0)$.

So that the N-S equation in cartesian form becomes:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

Since the flow is taken away x direction only and fully developed region, the continuity equation becomes $\frac{\partial u}{\partial x} = 0$, so that the N-S along x becomes:

$$\rho\left[\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial y^2}\right]$$

If in addition the flow is induced by constant pressure gradient there is not external force and zero suction / injection the equation reduces

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$ Divide by ρ.

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$ let $\nu = \frac{\mu}{\rho}$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \longrightarrow *$$

For steady state problem; $\frac{\partial u}{\partial t} = 0$ & since $\frac{\partial p}{\partial x}$ is constant, then the governing equation

is an ODE:

$$\frac{d^2 u}{dy^2} = \frac{1}{\nu\rho}\frac{dp}{dx} \longrightarrow (1)$$

Subject to:

$u = A$ at $y = 0$

$u = B$ at $y = H$

Integrate w.r.t y:

$$\frac{du}{dy} = \frac{y}{\nu\rho}\frac{dp}{dx} + c_1$$

Integrate again:

$$u(y) = \frac{y^2}{2\nu\rho}\frac{dP}{dx} + yC_1 + C_2.$$

when $u = A$ $\quad y = 0$

$\therefore A = C_2 : \quad C_2 = A.$

When $u = B \quad y = H.$

$$B = \frac{H^2}{2\nu\rho}\frac{dp}{dx} + HC_1 + A$$

$$C_1 = \frac{1}{H}\left[B - \frac{H^2}{2\nu\rho}\frac{dp}{dx} - A\right]$$

$$C_1 = \frac{B-A}{H} - \frac{H}{2\nu\rho}\frac{dp}{dx}$$

$\therefore$

$$u(y) = \frac{y^2}{2\nu\rho}\frac{dp}{dx} + y\left[\frac{B-A}{H} - \frac{H}{2\nu\rho}\frac{dp}{dx}\right] + A$$

$$u(y) = \frac{y^2}{2\nu\rho}\frac{dp}{dx} - \frac{Hy}{2\nu\rho}\frac{dp}{dx} + y\frac{(B-A)}{H} + A$$

$$u(y) = \frac{1}{2\nu\rho}\frac{dp}{dx}\left[y^2 - Hy\right] + \frac{(B-A)y + AH}{H}$$

This is Generalized cases of Navier-Stokes of 2-Dimensional.

Case I: Pressure Driven flow

Suppose the flow is strictly caused by constant pressure gradient only.

$A = 0$ & $B = 0$

We know that:

$$u(y) = \frac{1}{2\nu\rho}\frac{dp}{dx}(y^2 - Hy) + \frac{(B-A)y + AH}{H}$$

for $A = 0$ & $B = 0$, we have

$$u(y) = \frac{1}{2\nu\rho}\frac{dp}{dx}(y^2 - Hy) \longrightarrow \text{velocity}$$

To calculate the pressure gradient

$$\int_0^H u(y)\,dy = \int_0^H dy$$

$$\frac{1}{2\nu\rho}\frac{dp}{dx}\left[\frac{y^3}{3} - \frac{y^2 H}{2}\right]_0^H = H$$

$$\frac{dp}{dx}\frac{1}{2\nu\rho}\left(\frac{H^3}{3} - \frac{H^3}{2}\right) = H$$

$$-\frac{dp}{dx}\frac{1}{2\nu\rho}\left(\frac{H^3}{6}\right) = H$$

$$-\frac{dp}{dx} = \frac{12\nu\rho}{H^2}$$

$$\frac{dp}{dx} = \frac{-12\nu\rho}{H^2}$$

Skin friction

$$\left.\frac{du(y)}{dy}\right|_{y=0,H} = \frac{dP}{dx}\frac{1}{2\nu\rho}\left[2y-H\right]$$

$\therefore$

$y=0 \Rightarrow -\frac{dp}{dx}\frac{H}{2\nu\rho}$

$y=H \Rightarrow \frac{dp}{dx}\frac{H}{2\nu\rho}$.

Case II. Couette flow: $\frac{dp}{dx}=0$.

$\therefore u(y) = \frac{(B-A)y + AH}{H}$

i for $A\neq 0$, $B=1$

$\therefore$

$u(y) = y/H$.

ii. For $A=1$, $B=0$

$u(y) = \frac{-y+H}{H}$

iii for $A=-1$, $B=1$

$u(y) = \frac{2y-H}{H}$

Case III: Generalized Couette flow

This is a case where fluid motion is induced by combined pressure gradient to motion of atleast one of the plates. If the upper plate is fixed while the lower plate is moving:

$$U(y) = \frac{dp}{dx}\frac{1}{2\nu\rho}\left\{(y^2 - yH)\right\} + \frac{(B-A)y + AH}{H}$$

$\therefore A = 1$ $\&$ $B = 0$

That is

$$U(y) = \frac{dp}{dx}\frac{1}{2\nu\rho}\left\{\left[y^2 - yH\right] + \frac{H - y}{H}\right\}$$

Hegan - Poiseuille flow

Consider a steady flow of a viscous incompressible fluid is assumed to be laminar flow through very long straight pipe of circular cross section. The pipe is taken to be very long so that ends effect are negligible. Suppose the flow is along z-axis and that is no any external force acting on flow formation except by a constantly applied pressure in the z-direction. Discus the velocity, skin friction and induced pressure.

Soln

Navier-stokes along z:

$$\rho\left[\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right]$$

$$= f_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right]$$

$\because f_z = 0, \quad \frac{\partial u_z}{\partial t} = 0, \quad \frac{\partial^2 u_z}{\partial \theta^2} = 0, \quad \frac{\partial^2 u_z}{\partial z^2} = 0.$

$\therefore$ $\cancel{\frac{\partial u_z}{\partial r}}$:

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right]$$

$$\frac{\partial p}{\partial z} = \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right)$$

$$\frac{1}{\mu}\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

$$\frac{r}{\mu}\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

Integrate w.r.t r. we have

$$\frac{r^2}{2\mu}\frac{\partial p}{\partial z} = r\frac{\partial u_z}{\partial r} + c_1$$

or

$$r\frac{\partial u_z}{\partial r} = \frac{r^2}{2\mu}\frac{\partial p}{\partial z} + c_1 \qquad (1)$$

∴

$$\frac{\partial u_z}{\partial r} = \frac{r}{2\mu}\frac{\partial p}{\partial z} + \frac{c_1}{r}$$

Integrate again w.r.t to r, we have

$$u = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + c_1 \ln r + c_2 \qquad (2)$$

Case I : Pressure driven flow

Boundary Conditions:

$$\frac{du}{dr} = 0 \quad \text{at} \quad r=0$$

$$u=0 \quad \text{at} \quad r=a$$

Applying the first condition: on ①

$0 = 0 + C_1 \quad \therefore C_1 = 0.$

Applying the second condition on ②.

$$0 = \frac{a^2}{4\mu}\frac{\partial p}{\partial z} + 0 + C_2$$

$\therefore$

$$C_2 = -\frac{a^2}{4\mu}\frac{dp}{dz}$$

$\therefore$ The velocity along z becomes;

$$u_z = \frac{r^2}{4\mu}\frac{dp}{dz} + 0 + -\frac{a^2}{4\mu}\frac{dp}{dz}$$

$$u_z = \frac{(r^2 - a^2)}{4\mu}\frac{dp}{dz}$$

Case II: Couette flow : $\frac{dp}{dz} = 0.$

$\therefore$

$$\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right) = 0$$

Then: Eqn ① becomes:

$$r\frac{\partial u_z}{\partial r} = C_1$$

$\therefore$

$$\frac{du_z}{dr} = \frac{C_1}{r} \quad \text{—— ③}$$

Integrating ③ then we have

$$U_z = C_1 \ln r + C_2 \quad \text{——— ④}$$

Apply the condition:

$$\frac{du}{dr} = 0 \quad \text{at } r = 0.$$

from ③: $0 = \frac{C_1}{0} \Rightarrow C_1 = 0$

Apply the condition:

~~from ④~~:

$$U_z(r) = V_1 \quad \text{at } r = a.$$

∴ from 4:

$$V_1 = 0 \ln a + C_2$$

$$\therefore C_2 = V_1.$$

∴

$$U_z(r) = V_1$$

Case III; Generalized Couette flow.

$$\frac{du}{dr} = 0 \quad \text{at } r = 0$$

$$U_z(r) = V_1 \quad \text{at } r = a \qquad \therefore \frac{dp}{dz} \neq 0$$

The general solution

$$U_z(r) = \frac{r^2}{4\mu}\frac{dp}{dz} + C_1 \ln r + C_2$$

but $C_1 = 0$

$$U_z(r) = \frac{r^2}{4\mu}\frac{dp}{dz} + c_2.$$

Apply the condition $U_z(r) = V_1$ at $r = a$ we have:

$$V_1 = \frac{a^2}{4\mu}\frac{dp}{dr} + C_2$$

$$\therefore C_2 = V_1 - \frac{a^2}{4\mu}\frac{dp}{dz}$$

$\therefore$

$$U(r) = \frac{r^2}{4\mu}\frac{dp}{dr} + V_1 - \frac{a^2}{4\mu}\frac{dp}{dr}$$

$$U(r) = \left(\frac{r^2 - a^2}{4\mu}\right)\frac{dp}{dz} + V_1$$

Annus / Concentric Cylinder on Hegan-pouiville flow:

Case I: pressure driven

$u = 0$ at $r = a$

$u = 0$ at $r = b$

- The general solution:

$$u(y) = \frac{r^2}{4\mu}\frac{dp}{dz} + c_1 \ln r + c_2$$

- 1st condition:

$$\frac{a^2}{4\mu}\frac{dp}{dr} + c_1 \ln a + c_2 = 0 \quad \text{--- } 1$$

2nd condition:

$$\frac{b^2}{4\mu}\frac{dp}{dr} + c_1 \ln b + c_2 = 0 \quad \text{--- } 2$$

Subtract ② from ①:

$$\frac{(a^2 - b^2)}{4\mu}\frac{dp}{dr} + c_1 (\ln a - \ln b) = 0$$

$$\therefore c_1 (\ln a - \ln b) = \frac{(b^2 - a^2)}{4\mu}\frac{dp}{dz}$$

$$c_1 \ln a/b = \frac{(b^2 - a^2)}{4\mu}\frac{dp}{dz}$$

$\therefore$

$$c_1 = \frac{(b^2 - a^2)}{4\mu \ln a/b}\frac{dp}{dz}$$

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from ①: $C_2 = -C_1 \ln a - \frac{a^2}{4\mu}\frac{dp}{dz}$

$$C_2 = -\frac{\ln a \cdot (b^2 - a^2)}{4\mu \ln a/b}\frac{dp}{dr} - \frac{a^2}{4\mu}\frac{dp}{dz}$$

$$\cancel{C_2 = \left(\frac{-b^2 \ln a}{\ln a/b} - 1\right)\frac{a^2}{4\mu}\frac{dp}{dr}}$$

$$\cancel{C_2 = -\frac{(b^2 \ln a + \ln a/b)}{\ln a/b}\frac{a^2}{4\mu}\frac{dp}{dr}}$$

$$C_2 = \left(\frac{a^2 \ln a - b^2 \ln a}{4\mu \ln a/b} - a^2 \ln a/b\right)\frac{dp}{dz}$$

$$C_2 = \frac{(\cancel{a^2 \ln a} - b^2 \ln a - \cancel{a^2 \ln a} + a^2 \ln b)}{4\mu \ln a/b}\frac{dp}{dz}$$

$$C_2 = \frac{(a^2 \ln b - b^2 \ln a)}{4\mu \ln a/b}\frac{dp}{dz}.$$

$\therefore$

$$u_z(r) = \frac{r^2}{4\mu}\frac{dp}{dz} + \frac{b^2 - a^2}{4\mu \ln a/b}\ln r \frac{dp}{dz} + \frac{[a^2 \ln b - b^2 \ln a)}{4\mu \ln a/b}\frac{dp}{dz}$$

$\therefore$

$$u_z(r) = \frac{[r^2 \ln a/b + (b^2 - a^2)\ln r + (a^2 \ln b - b^2 \ln a)]}{4\mu \ln a/b}\frac{dp}{dz}$$

Case ⑪: Couette flow: $\therefore \frac{dP}{dz} = 0$

$u = V_1$ at $r = a$

$u = V_2$ at $r = b$.

$\therefore$ The general Eqn:

$$U(y) = C_1 \ln r + C_2$$

$\therefore$

1st condition:

$$V_1 = C_1 \ln a + C_2 \quad \text{--- } 1$$

$$V_2 = C_1 \ln b + C_2 \quad \text{--- } 2$$

substracting ② from ①,

$$V_1 - V_2 = C_1 (\ln a - \ln b)$$

$$C_1 = \frac{V_1 - V_2}{\ln a/b}$$

from ①: $C_2 = \cancel{V_2} V_1 - C_1 \ln a$

$$C_2 = V_1 - \frac{(V_1 - V_2)\ln a}{\ln a/b}$$

$$C_2 = \frac{V_1(\ln a - \ln b) - V_1 \ln a + V_2 \ln a}{\ln a/b}$$

$\therefore$

$$C_2 = \frac{V_2 \ln a - V_1 \ln b}{\ln a/b}$$

$$\therefore u_z(r) = \frac{(V_1 - V_2)}{\ln a/b} \ln r + \frac{V_2 \ln a - V_1 \ln b}{\ln a/b}$$

$$u_z(r) = \frac{V_1(\ln r - \ln b) - V_2(\ln r - \ln a)}{\ln a/b}$$

$$u(r) = \frac{V_1 \ln r/b - V_2 \ln r/a}{\ln a/b}$$

Case ~~III~~: Generalized Couette flow:

$\therefore \frac{dp}{dz} \neq 0.$

$u = V_1$ at $r = a$

$u = V_2$ at $r = b$

1. Stokes First and Second Problems

Consider an unsteady flow of viscous incompressible fluid past an infinite that start moving suddenly, with a uniform velocity U_0 on its own plane. Find the velocity and the skin friction at any point at time $t>0$:

$$u(y,0)=0$$

$$t>0 \begin{cases} u=U_0 & \text{at } y=0 \\ u \to 0 & \text{as } y \to \infty \end{cases}$$

The eqn is given by:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \text{where } \nu = \frac{\mu}{\rho}.$$

Take Laplace transform: we have

$$s\bar{u}(y,s) - u(y,0) = \nu \frac{d^2\bar{u}(y,s)}{dy^2}$$

$$s\bar{u}(y,s) = \nu \frac{d^2\bar{u}(y,s)}{dy^2} \quad \text{since } u(y,0)=0$$

$$\frac{d^2 \bar{u}(y,s)}{dy^2} = \frac{s}{\nu} \bar{u}(y,s)$$

$$\frac{d^2 \bar{u}(y,s)}{dy^2} - \frac{s}{\nu} \bar{u}(y,s) = 0 \quad \text{———} \ (*)$$

let $\bar{u}(y,s) = e^{my}$

$$\frac{d\bar{u}(y,s)}{dy} = m e^{my} \quad \therefore \frac{d^2 \bar{u}(y,s)}{dy^2} = m^2 e^{my}$$

Eqn $(*)$ becomes:

$$m^2 e^{my} - \frac{s}{\nu} e^{my} = 0$$

$$\left(m^2 - \frac{\nu}{s}\right) e^{my} = 0$$

$$\therefore m^2 - \frac{s}{\nu} = 0 \quad \text{since } e^{my} \neq 0.$$

$$m^2 = \frac{s}{\nu} \Rightarrow m = \pm\sqrt{s/\nu}$$

$\therefore$

$$\bar{u}(y,s) = C_1 e^{y\sqrt{s/\nu}} + C_2 e^{-y\sqrt{s/\nu}} \quad \text{———} \ (**)$$

Apply Condn 1: $u = u_0$

$$L(u) = \frac{u_0}{s}$$

$\therefore$

$$\bar{u}(y,s) = \frac{u_0}{s} \quad \text{at } y = 0$$

$\therefore$

$$\frac{u_0}{s} = C_1 + C_2 \quad \text{———} \ (1)$$

Apply condn 2: $u \to 0$ as $y \to \infty$

$\mathcal{L}(u) \Rightarrow 0$

$\bar{u}(y,s) \to 0$ as $y \to \infty$

$\therefore$

$$0 = c_1(\infty) + c_2(0)$$

$\therefore c_1 = 0$.

$\therefore$

~~$\bar{u}(y,s) =$~~ $\therefore$ $c_2 = \dfrac{u_0}{s}$.

$$\bar{u}(y,s) = \frac{u_0}{s} e^{-y\sqrt{s/\nu}} \quad \text{———} (***)$$

Take Laplace inverse:

$$\mathcal{L}^{-1}(\bar{u}(y,s)) = \mathcal{L}^{-1}\left(\frac{u_0}{s} e^{-y\sqrt{s/\nu}}\right)$$

$$u(y,t) = u_0 \, \mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

(ii) Skin friction:

$$\tau = -\mu \left.\frac{du}{dy}\right|_{y=0}$$

$$\tau = -\mu \left[\frac{u_0}{2\sqrt{\nu t}} e^{-\frac{y^2}{4\nu t}}\right]_{y=0}$$

$$\tau = \frac{-\mu U_0}{2\sqrt{\nu t}} e^0 \Rightarrow \tau = \frac{-\mu U_0}{2\sqrt{\nu t}}$$

Stokes Second problems

$$t > 0 \begin{cases} u = U_0 \cos\omega t \text{ at } y = 0 \\ u \to 0 \text{ as } y \to \infty \end{cases}$$

$t \leq 0, \; u = 0 \; \forall y.$

The periodic solution can be obtained by letting:

$$u(y,t) = U(y)\, e^{i\omega t}$$

The eqn is $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ where $\nu = \frac{\mu}{\rho}$.

$$\frac{\partial u}{\partial y} = U'(y)\, e^{i\omega t}$$

$$\frac{\partial^2 u}{\partial y^2} = U''(y)\, e^{i\omega t}$$

$$\frac{\partial u}{\partial t} = i\omega\, U(y)\, e^{i\omega t}$$

$$\therefore \; i\omega U \cancel{e^{i\omega t}} = \nu U'' \cancel{e^{i\omega t}}$$

$\Rightarrow u''(y) - i\omega u(y) = 0$

$u'' - \frac{i\omega}{\nu} u = 0$

$m^2 = \frac{i\omega}{\nu} \Rightarrow m = \pm\sqrt{\frac{i\omega}{\nu}}$

$m = \pm\sqrt{i}\sqrt{\omega/\nu} = \pm\frac{(1+i)}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}$.

$\therefore$

$u(y) = C_1 e^{\frac{(1+i)}{\sqrt{2}} y\sqrt{\omega/\nu}} + C_2 e^{-\frac{(1+i)}{\sqrt{2}} y\sqrt{\omega/\nu}}$ ——— *

Apply the Conditions: But

$U(y,t) = U_0 \cos\omega t$

$U(y,t) = U(y) e^{i\omega t} \Rightarrow u(y) = U_0$ at $y=0$

$\therefore$

$U_0 = C_1 e^0 + C_2 e^0$

$U_0 = C_1 + C_2$ ——— 1

Also, $u \to 0$ as $y \to \infty$

$0 = C_1(\infty) + C_2(0)$ $\therefore$

$C_1 = 0$.

$\therefore$

$\Rightarrow C_2 = U_0$

$\therefore u(y) = U_0 e^{-y\frac{(1+i)}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}}$ ——— **

So,

$u(y,t) = U(y) e^{i\omega t}$

$$u(y,t) = u_0 e^{-\frac{y(1+i)}{\sqrt{2}}\sqrt{\omega/\nu}} e^{i\omega t}$$

$$u(y,t) = u_0 e^{-\frac{y\sqrt{\omega/\nu}}{\sqrt{2}} - iy\frac{\sqrt{\omega/\nu}}{\sqrt{2}}} e^{i\omega t}$$

$$u(y,t) = u_0 e^{-y\sqrt{\frac{\omega}{2\nu}}} \cdot e^{-iy\sqrt{\frac{\omega}{2\nu}}} e^{i\omega t}$$

$$u(y,t) = u_0 e^{-y\sqrt{\frac{\omega}{2\nu}}} e^{-i\left(y\sqrt{\frac{\omega}{2\nu}} - \omega t\right)}$$

$$u(y,t) = u_0 e^{-y\sqrt{\frac{\omega}{2\nu}}} \left[\cos\left(y\sqrt{\tfrac{\omega}{2\nu}} - \omega t\right) - i\sin\left(y\sqrt{\tfrac{\omega}{2\nu}} - \omega t\right)\right]$$

We only consider the Real part:

$$u(y,t) = u_0 e^{-y\sqrt{\frac{\omega}{2\nu}}} \cos\left(y\sqrt{\tfrac{\omega}{2\nu}} - \omega t\right)$$

Another Case: For the periodic solution:

$$t > 0 \begin{cases} u = u_0 \sin \omega t \text{ at } y = 0 \\ u \longrightarrow 0 \text{ as } y \longrightarrow \infty \end{cases}$$

$$t \le 0, \quad u = 0 \quad \forall y$$

The periodic solution can be obtained by letting:

$$U(y,t) = U(y)\, e^{i\omega t}$$

The equation is: $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2}$ — (*)

$$\frac{\partial u}{\partial y} = u'(y)\, e^{i\omega t}$$

$$\frac{\partial^2 u}{\partial y^2} = u''(y)\, e^{i\omega t}$$

$$\frac{\partial u}{\partial t} = i\omega U(y)\, e^{i\omega t}$$

putting in (*):

$$i\omega U(y) e^{i\omega t} = \nu U''(y) e^{i\omega t}$$

$$i\omega U = \nu U''$$

$$\nu U'' - i\omega U = 0$$

$$U'' - \frac{i\omega}{\nu} U = 0$$

$\therefore$

$$m^2 - \frac{i\omega}{\nu} = 0$$

$$\sqrt{m^2} = \sqrt{\frac{i\omega}{\nu}} \Rightarrow m = \pm \sqrt{i}\sqrt{\frac{\omega}{\nu}}$$

$$m = \pm \frac{(1+i)}{\sqrt{2}} \sqrt{\frac{\omega}{\nu}}$$

$m_1 = +(1+i)\sqrt{\frac{w}{2\nu}}$, $m_2 = -(1+i)\sqrt{\frac{w}{2\nu}}$

$$U(y) = C_1 e^{(1+i)\sqrt{\frac{w}{2\nu}}\,y} + C_2 e^{-(1+i)\sqrt{\frac{w}{2\nu}}\,y} \quad \text{——} \;**$$

Apply the Conditions: But.

$U(y,t) = U_0 \cos wt$

$U(y,t) = U(y)e^{iwt} \Rightarrow U(y) = U_0$ at $y = 0$

$\therefore$

$U_0 = C_1 e^0 + C_2 e^0$

$U_0 = C_1 + C_2$ —— 1

$\therefore$

Also, $U(y,t) \to 0$ as $y \to \infty$

$0 = C_1(\infty) + C_2(0)$

$\therefore \Rightarrow C_1 = 0$

$U_0 = C_2$

The $(**)$ becomes:

$$U(y) = U_0 e^{-(1+i)y\sqrt{\frac{w}{2\nu}}}$$

But

$$U(y,t) = U(y)\, e^{iwt}$$

$$U(y,t) = U_0 e^{-(1+i)y\sqrt{\frac{w}{2\nu}}} \cdot e^{iwt}$$

$$U(y,t) = U_0 e^{-y\sqrt{\frac{w}{2\nu}}} \cdot e^{-iy\sqrt{\frac{w}{2\nu}}} \cdot e^{iwt}$$

$$U(y,t) = e^{-y\sqrt{\frac{w}{2\nu}}} \cdot e^{-i\left(y\sqrt{\frac{w}{2\nu}} - wt\right)}$$

$$U(y,t) = e^{-y\sqrt{\frac{w}{2\nu}}} \left[\cos\left(y\sqrt{\frac{w}{2\nu}} - wt\right) - i\sin\left(y\sqrt{\frac{w}{2\nu}} - wt\right)\right]$$

We only consider the imaginary part

$$\therefore U(y,t) = -e^{-y\sqrt{\frac{w}{2\nu}}} \sin\left(y\sqrt{\frac{w}{2\nu}} - wt\right)$$